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THE STABILITY OF THE GAUSS–CHEBYSHEV METHOD FOR CAUCHY SINGULAR INTEGRAL EQUATIONS

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Abstract—We show that the infinity condition number of the Gauss–Chebyshev method, for the complete Cauchy singular integral equation (CSIE) of the first kind, is bounded above and below by the condition number of the dominant equation times a constant. The condition number of the dominant equation is asymptotically equal to $\{(0.718 + 0.344 \ln n)n - 0.344\}$. This implies that the Gauss–Chebyshev method is stable for very large n's provided that the multiplying constants are not too large. The magnitude of the constants depends on the eigenvalues of the CSIE.

1. INTRODUCTION

In this paper we study the stability of the Gauss–Chebyshev numerical method for the solution of the Cauchy singular integral equation (CSIE),

$$\pi^{-1} \int_{-1}^{1} w(t) \frac{g(t)}{t-s} \,\mathrm{d}t + \lambda \int_{-1}^{1} w(t) K(t,s) g(t) \,\mathrm{d}t = f(s), \quad |s| < 1, \tag{1}$$

where $K(t, s)$ and $f(s)$ are given Hölder continuous functions and λ is a constant. The weight function $w(t)$ is determined by Nöether's index theory [1]. For simplicity let the index $\kappa = 1$, so that

$$w(t) = (1 - t^2)^{-1/2}. \tag{2}$$

For $\kappa = 1$, the solution is not unique. An additional condition must be used to ensure uniqueness. We assume that

$$\pi^{-1} \int_{-1}^{1} w(t) g(t) \,\mathrm{d}t = N, \tag{3}$$

where N is a given constant.

During the last decade several numerical methods have been developed for the solution of equation (1). Due mainly to the special form of the weight function $w(t)$, the numerical methods developed initially were based on orthogonal polynomial approximations with respect to $w(t)$ in $[-1, 1]$ of the input functions $K(t, s)$ and $f(s)$, (see Golberg [2] for a review). A widely used method for the solution of equation (1) is the Gauss–Chebyshev quadrature method [3]. Its popularity is due to the simplicity and high accuracy, particularly for problems with smooth input functions. For problems with non-smooth input functions other methods such as Galerkin and collocation will give better accuracy, provided that quadratures with high precision are used for the evaluation of certain integrals of the input functions [2]. If these integrals are approximated via Gauss-type quadratures then the Galerkin and colocation methods are mathematically equivalent to the quadrature method [4–9]. A similar equivalence result of the Galerkin and quadrature (or Nyström) methods for Fredholm integral equations (FIE) has recently been obtained by Atkinson and Bogolmolny [10]. However, even if these methods are mathematically equivalent the resulting algebraic systems are different and therefore they possess different stability characteristics.

The emphasis during the last decade has been in the study of the convergence of the previously mentioned methods [2]. We now know that they converge for a fairly large class of functions, but we have very little information about their stability. Fromme and Golberg [11] have studied the stability of the collocation method using a weighted $L^2_{1/w}$ norm. Linz [12] has suggested that his general theorems for operator equations could be used to study the stability of the Galerkin method. Srivastav [9] has obtained exact expressions for the ℓ_2 condition number of the coefficient matrix of the *dominant* CSIE, i.e. $K(t, s) = 0$, for the Gauss–Chebyshev quadrature method. However, since ℓ_∞ is the natural norm compatible with $C[-1, 1]$ the collectively compact operator theory cannot be used to extend Srivastav's stability results to the *complete* equation (1).

In the next section we describe the Gauss–Chebyshev quadrature method. We show that the condition number of the Gauss–Chebyshev method for the complete equation is bounded above and below by the condition number of the dominant equation times a constant. In subsection 2.1, we show that the condition number of the dominant equation is asymptotically equal to $\{(0.718 + 0.344 \ln n)n - 0.344\}$. This approximation gives us bounds for the condition number of the complete equation. It also implies that the Gauss–Chebyshev method is *stable* for very large ns provided that the multiplying constants are not too large. The magnitude of the multiplying constants depend on how close λ is to an eigenvalue of (1).

2. THE GAUSS-CHEBYSHEV QUADRATURE METHOD

We first derive the Gauss–Chebyshev method. By using the identity [3]

$$\pi^{-1} \int_{-1}^{1} w(t) \frac{1}{t-s} \, dt = 0, \quad |s| < 1, \tag{4}$$

we can "subtract" the singularity of equation (1) at $t = s$,

$$\pi^{-1} \int_{-1}^{1} w(t) \frac{g(t) - g(s)}{t - s} \, dt + \lambda \int_{-1}^{1} w(t) K(t, s) g(t) \, dt = f(s). \tag{5}$$

Approximating the integrals in expressions (5) and (3) via the Gauss–Chebyshev quadrature and using the identity [3]

$$n^{-1} \sum_{i=1}^{n} \frac{1}{t_i - s} = -\frac{U_{n-1}(s)}{T_n(s)}, \tag{6}$$

we obtain the functional equation

$$n^{-1} \sum_{i=1}^{n} \frac{g_n(t_i)}{t_i - s} + \frac{U_{n-1}(s)}{T_n(s)} g_n(s) + \lambda \pi n^{-1} \sum_{i=1}^{n} K(t_i, s) g_n(t_i) = f(s) \tag{7}$$

and

$$n^{-1} \sum_{i=1}^{n} g_n(t_i) = N, \tag{8}$$

where $t_i = \cos[(2i-1)\pi/2n]$, $i = 1, \ldots, n$, $T_n(s)$ and $U_{n-1}(s)$ are the Chebyshev polynomials of the first and second kind respectively, and $g_n(s)$ is an approximation to $g(s)$.

Collocating at $s_k = \cos(k\pi/n)$, $k = 1, \ldots, n-1$, in expression (7), we obtain the algebraic system

$$(\mathbf{A}_n + \lambda \mathbf{C}_n)\mathbf{g}_n = \mathbf{f}_n, \tag{9}$$

$$(\mathbf{A}_n)_{j,i} = \frac{1}{n(t_i - s_j)}, \quad (\mathbf{C}_n)_{j,i} = n^{-1}\pi K(t_i, s_j), \quad i = 1, \ldots, n, \quad j = 1, \ldots, n-1, \tag{10}$$

$$(\mathbf{A}_n)_{n,i} = n^{-1}, \quad (\mathbf{C}_n)_{n,i} = 0, \quad i = 1, \ldots, n, \tag{11}$$

where $\mathbf{g}_n = [g_n(t_1), \ldots, g_n(t_n)]^{\mathrm{T}}$ and $\mathbf{f}_n = [f(s_1), \ldots, f(s_{n-1}), N]^{\mathrm{T}}$. The matrix $\mathbf{A}_n$ possesses an inverse given by [5, 9]

$$(\mathbf{A}_n^{-1})_{i,j} = \frac{1 - s_j^2}{n(t_i - s_j)}, \quad (\mathbf{A}_n^{-1})_{i,n} = 1, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n-1.$$

Multiplying equation (9) by $\mathbf{A}_n^{-1}$ we obtain the algebraic system

$$(\mathbf{I}_n + \lambda \mathbf{Q}_n)\mathbf{g}_n = \mathbf{F}_n, \quad \mathbf{Q}_n = \mathbf{A}_n^{-1}\mathbf{C}_n, \quad \mathbf{F}_n = \mathbf{A}_n^{-1}\mathbf{f}_n, \tag{12}$$

where $\mathbf{I}_n$ is the unit matrix of order n. Let us define the kernel

$$L(x,t) = -\pi^{-1}\int_{-1}^{1}[w(s)]^{-1}g(x,t,s)\,\mathrm{d}s + tK(x,t),$$

$$g(x,t,s) = \frac{K(s,x)-K(t,x)}{s-t}, \tag{13}$$

and a sequence approximating $L(x,t)$,

$$L_n(x,t) = -n^{-1}\sum_{k=1}^{n-1}(1-s_k^2)g(x,t,s_k) + tK(t,x). \tag{14}$$

It can be shown that the matrix $\mathbf{Q}_n$ is identical to (see Refs [5, 13] for details)

$$(\mathbf{Q}_n)_{i,j} = n^{-1}L_n(t_j,t_i), \quad i,j = 1,\ldots,n. \tag{15}$$

The algebraic systems (9) and (12) provide numerical approximations to the solution of equation (1) at the points t_i, $i = 1,\ldots,n$. To find the solution for points different than t_i, we can use the natural interpolation formula (7) [14]. It has been shown in equation (15) that the natural interpolation formula converges uniformly to the solution of equation (1) provided that $K(t,s)$ and $f(s)$ are C^1 functions.

We now analyze the stability of the algebraic systems (9) and (12). We need to derive bounds for the *infinity* condition numbers

$$\kappa_\infty(\mathbf{A}_n + \lambda\mathbf{C}_n) = \|\mathbf{A}_n + \lambda\mathbf{C}_n\|_\infty \|(\mathbf{A}_n + \lambda\mathbf{C}_n)^{-1}\|_\infty \tag{16}$$

and $\kappa_\infty(\mathbf{I}_n + \lambda\mathbf{Q}_n)$, where the infinity norm of a matrix $\mathbf{B}$ with elements $b_{i,j}$ is defined by

$$\|\mathbf{B}\|_\infty = \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|b_{i,j}|. \tag{17}$$

We summarize the results of this section in the following theorem.

Theorem 1. If $K(t,s)$ and $f(s)$ are C^1 functions, then there exists an integer n_0 such that for $n \geqslant n_0$

$$\kappa_\infty(\mathbf{I}_n + \lambda\mathbf{Q}_n) < B(\lambda), \tag{18}$$

$$B_1(\lambda)\kappa_\infty(\mathbf{A}_n) \leqslant \kappa_\infty(\mathbf{A}_n + \lambda\mathbf{C}_n) \leqslant B(\lambda)\kappa_\infty(\mathbf{A}_n), \tag{19}$$

where $B(\lambda)$ and $B_1(\lambda)$ are constants independent of n.

Proof: the bound for inequality (*18*). From [15], we have that

$$\|(\mathbf{I}_n + \lambda\mathbf{Q}_n)^{-1}\|_\infty \leqslant B_2(\lambda), \quad n \geqslant n_0, \tag{20}$$

where $B_2(\lambda)$ is a constant independent of n. Furthermore,

$$\|\mathbf{I}_n + \lambda\mathbf{Q}_n\|_\infty \leqslant 1 + |\lambda| n^{-1}\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|L_n(t_j,t_i)| \leqslant 1 + |\lambda| M, \tag{21}$$

where

$$M = \max_{x,t}|L(x,t)|$$

is finite, because of expression (13) and the fact that $K(x,t)$ is a C^1 function. By combining the last two equations we derive

$$\kappa_\infty(\mathbf{I}_n + \lambda\mathbf{Q}_n) \leqslant B(\lambda), \quad B(\lambda) = (1 + |\lambda| M)B_2(\lambda). \tag{22}$$

Proof: the bounds for inequality (*19*). For the upper bound we use the fact that $\mathbf{Q}_n = \mathbf{A}_n^{-1}\mathbf{C}_n$ to show that

$$\kappa_\infty(\mathbf{A}_n + \lambda\mathbf{C}_n) \leqslant \kappa_\infty(\mathbf{A}_n)\kappa_\infty(\mathbf{I}_n + \lambda\mathbf{Q}_n). \tag{23}$$

For the lower bound, we first consider the quotient

$$\frac{\|(\mathbf{A}_n+\lambda\mathbf{C}_n)^{-1}\|_\infty}{\|\mathbf{A}_n^{-1}\|_\infty}=\frac{\|(\mathbf{A}_n+\lambda\mathbf{C}_n)^{-1}\|_\infty}{\|(\mathbf{I}_n+\lambda\mathbf{A}_n^{-1}\mathbf{C}_n)(\mathbf{A}_n+\lambda\mathbf{C}_n)^{-1}\|_\infty}\geqslant\frac{1}{1+|\lambda|M}, \tag{24}$$

to show that

$$\|(\mathbf{A}_n+\lambda\mathbf{C}_n)^{-1}\|_\infty\geqslant\frac{\|\mathbf{A}_n^{-1}\|_\infty}{1+|\lambda|M},\quad n\geqslant n_0, \tag{25}$$

where again we have used $\mathbf{Q}_n=\mathbf{A}_n^{-1}\mathbf{C}_n$ and expression (21).

Next, we consider

$$\|\mathbf{A}_n+\lambda\mathbf{C}_n\|_\infty\geqslant|(\|\mathbf{A}_n\|_\infty-|\lambda|\,\|\mathbf{C}_n\|_\infty)|=\|\mathbf{A}_n\|_\infty\left(\left|1-|\lambda|\frac{\|\mathbf{C}_n\|_\infty}{\|\mathbf{A}_n\|_\infty}\right|\right). \tag{26}$$

To derive a lower bound for the last equation we need to use the following asymptotic expansion for $\|\mathbf{A}_n\|_\infty$:

$$\|\mathbf{A}_n\|_\infty=C_0n+\frac{C_1}{n}+\frac{C_1^n}{n^3},$$
$$C_0\simeq 0.54,\quad C_1\simeq 0.56,\quad 0<C_1^n\leqslant 1.0, \tag{27}$$

whose proof we delay until Theorem 2(i). The use of n as a superscript in C_1^n, and in all other constants defined henceforth, signifies the dependence of this constant on n. We can easily see that there exist an integer n_0 such that

$$1-|\lambda|\frac{\|\mathbf{C}_n\|_\infty}{\|\mathbf{A}_n\|_\infty}\geqslant 1-|\lambda|\frac{\max_{x,t}|K(x,t)|}{\|\mathbf{A}_n\|_\infty}\geqslant\frac{1}{2},\quad n\geqslant n_0, \tag{28}$$

because of expression (27) and the fact that $K(t,s)$ is a C^1 function. Therefore, equation (26) is reduced to

$$\|\mathbf{A}_n+\lambda\mathbf{C}_n\|_\infty\geqslant\frac{\|\mathbf{A}_n\|_\infty}{2}. \tag{29}$$

Finally, by combining expressions (29) and (25) we obtain

$$\kappa_\infty(\mathbf{A}_n+\lambda\mathbf{C}_n)\geqslant B_1(\lambda)\kappa_\infty(\mathbf{A}_n),\quad B_1(\lambda)=\frac{1}{2(1+|\lambda|M)}.\quad\blacksquare \tag{30}$$

The last theorem implies that the stability of system (12) depends on the magnitude of the constant $B(\lambda)$, while the stability of system (9) depends on $B(\lambda)$, $B_1(\lambda)$ and also on the condition number $\kappa_\infty(\mathbf{A}_n)$ of the dominant equation. Following Atkinson [16, p. 105], we can say that system (12) is stable provided that $B(\lambda)$ is not too large. However, for the stability of system (9) we need to determine the magnitude for $\kappa_\infty(\mathbf{A}_n)$ which we do in the next subsection.

2.1. The Condition Number for the Dominant Equation

In this subsection we derive approximations for $\|\mathbf{A}_n\|_\infty$ and $\|\mathbf{A}_n^{-1}\|_\infty$. We use these approximations to analyze the stability of the Gauss–Chebyshev method.

Let us define

$$S(i)=n^{-1}\sum_{j=1}^{n}\frac{1}{|t_j-s_i|},\quad i=1,\ldots,n-1,$$

$$R(j)=n^{-1}\sum_{k=1}^{n-1}\frac{1-s_k^2}{|t_j-s_k|},\quad j=1,\ldots,n. \tag{31}$$

By using the identities [3]

$$n^{-1}\sum_{j=1}^{n}\frac{1}{t_j-s_i}=0,\quad i=1,\ldots,n-1,$$

$$n^{-1}\sum_{k=1}^{n-1}\frac{1-s_k^2}{t_j-s_k}=t_j,\quad j=1,\ldots,n, \tag{32}$$

and the fact that

$$-1<t_n<s_{n-1}<\cdots<s_k<t_k<\cdots<s_1<t_1<1, \tag{33}$$

we obtain

$$S(i)=2n^{-1}\sum_{j=1}^{i}\frac{1}{t_j-s_i},\quad i=1,\ldots,n-1,$$

$$R(j)=2n^{-1}\sum_{k=1}^{j-1}\frac{1-s_k^2}{s_k-t_j}+t_j,\quad j=1,\ldots,n. \tag{34}$$

Since $t_{n-j+1}=-t_j$ and $s_{n-i}=-s_i$, we can easily see that $S(i)=S(n-i)$, $i=1,\ldots,\lfloor n/2\rfloor$, and $R(j)=R(n-j+1), j=1,\ldots,\lceil n/2\rceil$. We have strong numerical evidence that the sequences $S(i)$ and $R(j)$ satisfy the inequalities $S(1)>S(2)>\cdots>S(\lfloor n/2\rfloor)$ and $R(1)<R(2)<\cdots<R(\lceil n/2\rceil)$. However, we have not been able to provide a formal proof and therefore we present the following results as a Conjecture.

Conjecture.

(a) $\max_{1\leqslant i\leqslant n-1} S(i)=S(1)=S(n-1)$ and (b) $\max_{1\leqslant j\leqslant n} R(j)=R(\lceil n/2\rceil)$.

We are now ready to derive expressions for $\|\mathbf{A}_n\|_\infty$ and $\|\mathbf{A}_n^{-1}\|_\infty$.

Theorem 2. If the Conjecture is true, then for all $n\geqslant 3$:

(i) $$\|\mathbf{A}_n\|_\infty=\frac{1}{n\sin\left(\dfrac{\pi}{4n}\right)\sin\left(\dfrac{3\pi}{4n}\right)}=C_0n+\frac{C_1}{n}+C_1^nh^3,$$

where $h=\pi/n$, $C_0=16/3\pi^2\simeq 0.54$, $C_1=10/18\simeq 0.56$ and where the constant C_1^n depends on n and satisfies $0\leqslant C_1^n\leqslant 1.0$;

(ii) $$\|\mathbf{A}_n^{-1}\|_\infty=C_2+C_3\ln(\lceil n/2\rceil]-\frac{C_3}{n}+C_2^nh^2,$$

where $C_2=2\pi^{-1}\{\gamma-1+\pi/2+\ln(16/\pi)\}\simeq 1.77$, $C_3=2/\pi\simeq 0.637$ and where C_2^n depends on n and satisfies $|C_2^n|\leqslant 6$.

Proof (i). The Conjecture implies that

$$\|\mathbf{A}_n\|_\infty=\max\left\{\max_{1\leqslant i\leqslant n-1} S(i),1\right\}=2n^{-1}\sum_{j=1}^{n-1}\frac{1}{t_j-s_i}=\frac{2}{n(s_{n-1}-t_n)}. \tag{35}$$

Substituting $t_n=\cos[(2n-1)\pi/2n]$ and $s_{n-1}=\cos[(n-1)\pi/n]$ we obtain

$$\|\mathbf{A}_n\|_\infty=\frac{1}{n\sin\left(\dfrac{\pi}{4n}\right)\sin\left(\dfrac{3\pi}{4n}\right)}.$$

Finally, by expanding $\sin(x)$ as $x-x^3/3!+x^5\cos(\xi)/5!$ we can easily show that

$$\frac{C_1^n}{n^3}=\frac{1}{n\sin\left(\dfrac{\pi}{4n}\right)\sin\left(\dfrac{3\pi}{4n}\right)}-C_0n-\frac{C_1}{n}, \tag{36}$$

where $C_0 = 16/3\pi^2 \simeq 0.54$, $C_1 = 10/18 \simeq 0.56$ and where the constant C_1^n depends on n and satisfies $0 \leqslant C_1^n \leqslant 1.0$ and

$$\lim_{n \to \infty} C_1^n = (16/3\pi)[(10/96)^2 - 0.0036]\pi^3 \simeq 0.38.$$

Proof (ii). From the Conjecture and the definition of $\mathbf{A}_n^{-1}$ we have that

$$\| \mathbf{A}_n^{-1} \|_\infty = \max_{1 \leqslant j \leqslant n} R(j) + 1 = R(p) + 1, \tag{37}$$

where $p = \lceil n/2 \rceil$. We will use the trapezoidal rule

$$h \sum_{j=0}^{m} f(x_j) = \int_a^b f(x)\,\mathrm{d}x + \frac{h}{2}\{f(a) + f(b)\} + \frac{(b-a)h^2}{12} f''(\xi), \tag{38}$$

where $h = (b-a)/m$, $x_j = a + jh$, $j = 0, \ldots, m$ and $a < \xi < b$, to derive an asymptotic approximation for $R(p)$. We set $h = \pi/n$ and

$$\alpha = \begin{cases} \pi/2 & \text{if } n \text{ is odd} \\ \pi/2 - h/2 & \text{if } n \text{ is even} \end{cases} \tag{39}$$

so that,

$$t_p = \cos(\alpha). \tag{40}$$

We rewrite $R(p)$ as

$$R(p) = R_1(p) + R_2(p) + t_p, \tag{41}$$

$$R_1(p) = 2\pi^{-1} h \sum_{k=0}^{p-1} \left[\frac{1 - \cos^2(kh)}{\cos(kh) - t_p} - \frac{\sin(\alpha)}{\alpha - kh} \right],$$

$$R_2(p) = 2\pi^{-1} h \sin(\alpha) \sum_{k=0}^{p-1} \frac{1}{\alpha - kh}. \tag{42}$$

An approximation for $R_1(p)$

We set $m = p - 1$, $a = 0$, $b = (p-1)h$ and

$$f(x) = 2\pi^{-1} \left[\frac{1 - \cos^2(x)}{\cos(x) - t_p} - \frac{\sin(\alpha)}{\alpha - x} \right]$$

in expression (38), to obtain

$$R_1(p) = \int_0^{(p-1)h} f(x)\,\mathrm{d}x + \frac{h}{2}[f(a) + f(b)] + D_1^n h^2,$$

$$D_1^n = \frac{(b-a) f''(\xi)}{12}, \tag{43}$$

where (see Gradshteyn and Ryznik [17, p. 148] for the integration formulae)

$$\frac{h}{2}[f(a) + f(b)] \doteq \pi^{-1} h \left\{ \frac{1 - \cos^2[(p-1)h]}{\cos[(p-1)h] - t_p} - \frac{\sin(\alpha)}{\alpha - (p-1)h} - \frac{\sin(\alpha)}{\alpha} \right\}, \tag{44}$$

$$\int_0^{(p-1)h} f(x)\,\mathrm{d}x = 2\pi^{-1}\{I_3 - I_2\}, \tag{45}$$

$$I_1 = \int_0^{(p-1)h} \frac{\mathrm{d}x}{\cos(x) - t_p}$$

$$= (1 - t_p^2)^{-1/2} \ln \left\{ \frac{|1 - t_p + (1 - t_p^2)^{1/2} \tan[(p-1)h/2]|}{|1 - t_p - (1 - t_p^2)^{1/2} \tan[(p-1)h/2]|} \right\}, \tag{46}$$

$$I_2 = \int_0^{(p-1)h} \frac{\mathrm{d}x}{\alpha - x} = -\ln \left[\frac{|(p-1)h - \alpha|}{|\alpha|} \right] \tag{47}$$

and

$$I_3=\int_0^{(p-1)h}\frac{1-\cos^2(x)}{\cos(x)-t_p}\,dx=(1-t_p^2)I_1-(p-1)ht_p-\sin[(p-1)h]. \tag{48}$$

Since the magnitudes of the constants bounding the condition number are important in the analysis of its stability, we will derive a bound for D_1^n. To do that we need to bound

$$f''(x)=2\pi^{-1}\left\{\cos(x)+\frac{\sin^2(\alpha)[1+\sin^2(x)-\cos(x)\cos(\alpha)]}{[\cos(x)-\cos(\alpha)]^3}-\frac{2\sin(\alpha)}{(\alpha-x)^3}\right\}. \tag{49}$$

It can be shown that $f''(x)$ is a monotone decreasing positive function in $[0,(p-1)h]$, which implies that the maximum of $f''(x)$ in $[0,(p-1)h]$ and a bound for D_1^n are given by

$$f''(0)=2\pi^{-1}\left\{1+\frac{\sin^2(\alpha)}{[1-\cos(\alpha)]^2}-\frac{2\sin(\alpha)}{\alpha^3}\right\},\quad 0\leqslant D_1^n\leqslant\frac{(p-1)hf''(0)}{12}. \tag{50}$$

We will further simplify the expression for $R_1(p)$. We consider two special cases of n.

The case of odd n. For odd n's we have $\alpha=\pi/2$, $p=(n+1)/2$ and $t_p=0$. Using equation (50) we obtain

$$D_1^n\leqslant\frac{1}{12}\left(\frac{n+1}{2}-1\right)\frac{\pi}{n}\frac{4}{\pi}\left(1-\frac{8}{\pi^3}\right)\leqslant 0.08,\quad \text{for}\quad n\geqslant 3. \tag{51}$$

Next we consider equation (44). Expanding $\cos(x)=1-x^2\cos(\xi_1)/2$ and $\sin(x)=x-x^3\cos(\xi_2)/6$, we derive

$$\frac{h}{2}[f(a)+f(b)]=\pi^{-1}h\left[\frac{\cos^2(h/2)}{\sin(h/2)}-\frac{2}{h}-\frac{2}{\pi}\right]=-\frac{2h}{\pi^2}+D_2^nh^2 \tag{52}$$

and

$$D_2^n=2\pi^{-1}\left\{\frac{\cos(\xi_2)/24-\cos(\xi_1)[1-h^2\cos(\xi_1)/16]/4}{1-h^2\cos(\xi_2)/24}\right\}, \tag{53}$$

where $0\leqslant\xi_1,\ \xi_2\leqslant h/2$. We can easily see that

$$\lim_{n\to\infty}D_2^n=-10/24\pi\simeq-0.1326$$

and that

$$\begin{aligned}|D_2^n|&\leqslant 2\pi^{-1}\left[\frac{1/24+(1+h^2/16)/4}{1-h^2/24}\right]\\&\leqslant 2\pi^{-1}\left[\frac{1/24+(1+1.102/16)/4}{1-1.102/24}\right]\simeq 0.2061,\end{aligned} \tag{54}$$

where we have used the fact that for $n\geqslant 3$ the stepsize $h=\pi/n\leqslant 1.05$.

Similarly, for equations (45)–(48) we have that

$$I_2=\ln\left(\frac{\pi}{h}\right),\quad I_3=-\ln\left[\tan\left(\frac{h}{4}\right)\right]-\cos\left(\frac{h}{2}\right), \tag{55}$$

$$2\pi^{-1}\{I_3-I_2\}=-2\pi^{-1}\left\{\ln\left[\frac{\tan(z)}{z}\right]+\cos(h/2)+\ln(\pi/4)\right\},\quad z=h/4, \tag{56}$$

and by expanding $\tan(z)=z+z^3C(\xi)/3$, $C(\xi)=[1+2\sin^2(\xi)]/\cos^4(\xi)$ and $\ln(1+x)=x-x^2/2(1+\xi)^2$, where $0\leqslant\xi\leqslant h/2$, we derive

$$2\pi^{-1}\{I_3-I_2\}=-2\pi^{-1}\left[\ln\left(\frac{\pi}{4}\right)+1\right]+D_3^nh^2, \tag{57}$$

where the asymptotic limit and an upper bound for D_3^n are given by

$$\lim_{n\to\infty}D_3^n=0.0663,\quad |D_3^n|\leqslant 0.1326. \tag{58}$$

Finally, by combining expressions (52), (56) and (43) we derive

$$R_1(p) = -2\pi^{-1}\left[\ln\left(\frac{\pi}{4}\right) + 1 + \frac{1}{n}\right] + D_4^n h^2, \quad |D_4^n| \leqslant 1, \quad \text{for} \quad n \geqslant 3. \tag{59}$$

The case of even n. For even n's we have $p = n/2$, $t_p = \sin(h/2)$, and

$$D_1^n \leqslant 0.3794, \quad \text{for} \quad n \geqslant 3, \tag{60}$$

$$\frac{h}{2}[f(a) + f(b)] = -\frac{2h}{\pi(\pi - h)} + D_2^n h^2, \quad |D_2^n| \leqslant 1.8792, \quad \lim_{n\to\infty} D_2^n = -0.3714, \tag{61}$$

$$I_2 = \ln\left(\frac{\pi - h}{h}\right), \quad I_3 = \cos(h/2)\ln\left(\frac{c+d}{c-d}\right) - \left(\frac{\pi}{2} - h\right)\sin(h/2) - \cos(h), \tag{62}$$

where

$$c = 1 - \sin(h/2), \quad d = \cos(h/2)\tan[(\pi - 2h)/4]. \tag{63}$$

Using similar expansions as in the odd case for $\sin(x)$, $\cos(x)$ and $\ln(1 + x)$, we derive

$$2\pi^{-1}\{I_3 - I_2\} = -2\pi^{-1}\left[\ln\left(\frac{\pi - h}{4}\right) + 1 + \frac{\pi h}{4}\right] + [D_3^n + D_1 \ln(h)]h^2, \tag{64}$$

where

$$\lim_{n\to\infty} D_3^n = 0.2544, \quad |D_3^n| \leqslant 2, \quad D_1 = 1/4\pi = 0.0796. \tag{65}$$

Combining expression (64), (61) and (42), we obtain

$$R_1(p) = -2\pi^{-1}\left[\ln\left(\frac{\pi - h}{4}\right) + 1 + \frac{h}{\pi - h}\right] + [D_4^n + D_1 \ln(h)]h^2, \quad |D_4^n| \leqslant 4. \tag{66}$$

An approximation for $R_2(p)$

We consider again the special cases of even and odd n's.

The case of odd n. We have $p = (n + 1)/2$, $t_p = 0$ and if we replace $h = \pi/n$ and $\sin(\alpha) = 1$ in the second equation of (42), we derive

$$R_2(p) = 4\pi^{-1}\sum_{k=0}^{p-1}\frac{1}{n - 2k} = 4\pi^{-1}\sum_{k=1}^{p}\frac{1}{2k - 1} = 4\pi^{-1}$$
$$\times\left\{\frac{1}{2}\left[\gamma + \ln\left(\frac{n+1}{2}\right)\right] + \ln(2)\right\} + E_1^n h^2, \tag{67}$$

where $\gamma = 0.5772\ldots$ is Euler's constant and

$$E_1^n h^2 = 4\pi^{-1}\left\{\frac{B_2 h^2}{8\pi^2 a_n^2} + \frac{(2^3 - 1)B_4 h^4}{64\pi^4 a_n^4} + \cdots\right\}, \quad a_n = 1 + \frac{1}{n},$$

where $B_2 = 1/6$, $B_4 = -1/30, \ldots,$ are Bernouli's numbers [17]. By using the well-known fact for alternating series that the absolute value of the remainder does not exceed the absolute value of the first neglected term [18], we can show that

$$|E_1^n| \leqslant 0.0183. \tag{68}$$

The case of even n. For even n's we have $p = n/2$, $\sin(\alpha) = \cos(h/2)$ and

$$R_2(p) = 4\pi^{-1}\cos(h/2)\sum_{k=0}^{p-1}\frac{1}{n - (2k+1)} = 4\pi^{-1}\cos(h/2)\sum_{k=1}^{p}\frac{1}{2k-1}$$
$$= 4\pi^{-1}\left\{\frac{1}{2}\left[\gamma + \ln\left(\frac{n}{2}\right)\right] + \ln(2)\right\} + [E_2^n + E_1 \ln(h)]h^2, \tag{69}$$

where

$$|E_2^n| \leqslant 0.2105, \quad E_1 = -D_1 = -0.0796.$$

Table 1. Bounds for the condition number of the Gauss–Chebyshev method

n	$B_1^n(\lambda)$	$B_1^n(\lambda)\kappa_\infty(\mathbf{A}_n)$	$\kappa_\infty(\mathbf{A}_n+\lambda\mathbf{C}_n)$	$B^n(\lambda)$	$B^n(\lambda)\kappa_\infty(\mathbf{A}_n)$
			$\lambda=1$		
10	0.2119	0.3210×10^1	0.1675×10^2	0.4574×10^1	0.6931×10^2
20	0.2721	0.9516×10^1	0.3592×10^2	0.4958×10^1	0.1734×10^3
40	0.3026	0.2403×10^2	0.7998×10^2	0.5148×10^1	0.4089×10^3
			$\lambda=-0.6366$		
10	0.4090	0.6197×10^1	0.1930×10^6	0.7613×10^5	0.1151×10^7
20	0.4259	0.1489×10^2	0.3570×10^6	0.8476×10^5	0.2960×10^7
40	0.4332	0.3440×10^2	0.7000×10^6	0.8902×10^5	0.7070×10^7

Finally, by substituting $R_1(p)$ and $R_2(p)$ in expression (37) we obtain

$$\|\mathbf{A}_n^{-1}\|_\infty = R_1(p)+R_2(p)+t_p+1 = C_2+C_3\ln(\lceil n/2\rceil)-\frac{C_3}{n}+C_2^n h^2$$

$$C_2=2\pi^{-1}[\gamma-1+\pi/2+\ln(16/\pi)]\simeq 1.77,\quad C_3=2/\pi\simeq 0.637,\quad |C_2^n|\leqslant 6, \tag{70}$$

where for the even case we have rewritten $\ln[16/(\pi-h)]=\ln(16/\pi)-\ln(1-1/n)$ and expanded $\ln(1-1/n)$ and also $1/(1-1/n)$ in their power series. ∎

The following $O(h\ln h)$ approximation to $\kappa_\infty(\mathbf{A}_n)$ can be derived directly from the last theorem.

Corollary. $\kappa_\infty(\mathbf{A}_n)\approx(0.718+0.344\ln n)n-0.344$.

From the above Corollary we see that the condition number of the dominant equation grows slowly with n. For $n=40$ we have $\kappa_\infty(\mathbf{A}_n)\approx 79.13$ compared to the "exact" 79.43 obtained via MATLAB [19]. Therefore, the inequalities (18) and (19) imply that the Gauss–Chebyshev method is stable for very large n's, provided that the constant $B(\lambda)$ is small. How small is the constant $B(\lambda)$ depends on how close is λ to an eigenvalue of (1) [16]. The following example shows the effect of λ being close to an eigenvalue on the condition number of the Gauss–Chebyshev method.

Example. We consider equations (1) and (3) with

$$K(t,s)=t,\quad N=0. \tag{71}$$

We can easily see that an eigenvalue–eigenvector pair of

$$\pi^{-1}⨍_{-1}^{1} w(t)\frac{g(t)}{t-s}\,\mathrm{d}t+\lambda\int_{-1}^{1} w(t)tg(t)\,\mathrm{d}t=0,\quad |s|<1, \tag{72}$$

is given by

$$\lambda=-2/\pi=-0.6366\ldots,\quad g(t)=t.$$

In Table 1 the quantities

$$B_1^n(\lambda)=\left(\left|1-|\lambda|\frac{\|\mathbf{C}_n\|_\infty}{\|\mathbf{A}_n\|_\infty}\right|\right)\Big/\|\mathbf{I}+\lambda\mathbf{Q}_n\|_\infty\geqslant B_1(\lambda)=\frac{1}{2(1+2|\lambda|)} \tag{73}$$

and

$$B^n(\lambda)=\kappa_\infty(\mathbf{I}+\lambda\mathbf{Q}_n)\leqslant B(\lambda) \tag{74}$$

have been computed instead of $B_1(\lambda)$ and $B(\lambda)$. All computations have been performed in a DEC-20, using the interactive program MATLAB. As expected, both $B_1^n(\lambda)$ and $B^n(\lambda)$ converge to a constant as n increases. Also notice, that for $\lambda=1$, $B_1^n(\lambda)\geqslant B_1(\lambda)=1/6$, since $L(x,t)=-1+xt$, and

$$M=\max_{x,t}|L(x,t)|=2.$$

The upper bound $B^n(\lambda)\kappa_\infty(\mathbf{A}_n)$ of the condition number $\kappa_\infty(\mathbf{A}_n+\lambda\mathbf{C}_n)$ appears to be a sharper estimate than the lower bound $B_1^n(\lambda)\kappa_\infty(\mathbf{A}_n)$.

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